The three-dimensional noncommutative Gross–Neveu model

2007 J. Phys. A: Math. Theor. 40 3633

(http://iopscience.iop.org/1751-8121/40/13/020)

J. Phys. A: Math. Theor. **40** (2007) 3633–3641 doi:10.1088/1751-8113/40/13/020

The three-dimensional noncommutative Gross–Neveu model

B Charneski, A F Ferrari and M Gomes

Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970, São Paulo, SP, Brazil

E-mail: bruno@fma.if.usp.br, alysson@fma.if.usp.br and mgomes@fma.if.usp.br

Received 20 November 2006, in final form 18 January 2007
Published 14 March 2007
Online at stacks.iop.org/JPhysA/40/3633

Abstract
This work is dedicated to the study of the noncommutative Gross–Neveu model. As is known, in the canonical Weyl–Moyal approach the model is inconsistent, basically due to the separation of the amplitudes into planar and nonplanar parts. We prove that if instead a coherent basis representation is used, the model becomes renormalizable and free of the aforementioned difficulty. We also show that, although the coherent state procedure breaks Lorentz symmetry in odd dimensions, in the Gross–Neveu model, this breaking can be kept under control by assuming the noncommutativity parameters to be small enough. We also make some remarks on ordering prescriptions used in the literature.

PACS numbers: 11.10.Nx, 11.10.Gh, 11.10.Kk

1. Introduction

In recent years much effort has been devoted to the study of noncommutative field theories [1]. One important outcome of these investigations is that, for the case of canonical noncommutativity, the use of the Weyl–Moyal correspondence leads to strong nonlocal effects, which put severe restrictions on the form of the allowed models. In fact, it has been found that part of the ultraviolet divergences of the commutative models are transmuted into infrared ones. Whenever they are stronger than logarithmic, these divergences, called ultraviolet/infrared (UV/IR) singularities, are very dangerous, leading to a breakdown of most of the perturbative schemes. Even when the UV/IR infrared singularities are only logarithmic, the mere separation of contributions into planar (UV divergent) and nonplanar (UV finite but divergent whenever the external momenta tend to zero) parts, typical of the Weyl–Moyal method, may lead to inconsistencies in the renormalization program so that the model under scrutiny becomes nonrenormalizable. Examples where such situation occurs are provided by the four-dimensional $O(N)$ linear sigma model with $N > 2$ [2] and the $1/N$ expansion of the $O(N)$ Gross–Neveu (GN) model in 2 + 1 dimensions [3, 4]. In both cases the feature responsible for the failure of the renormalization procedure is the existence of a parameter whose renormalization in the commutative setting secures the elimination of the UV divergence of two different structures. For the linear sigma model it is the pion mass counterterm which enforces both the vanishing of the pion mass and the finiteness of the gap equation. In the GN model the coupling constant renormalization plays a double role enforcing the gap equation and also eliminating the UV divergence in the two-point vertex function of the auxiliary field introduced to implement the $1/N$ expansion. It was proved that enlarging the models, specifically, the gauging of the linear sigma model and the supersymmetrization of the GN model, furnished consistent theories without the aforementioned difficulty.

In the present work we will investigate an alternative procedure to introduce noncommutativity in field theories aiming at the construction of a consistent GN model without the necessity of supersymmetrization. More precisely, we will analyse a coherent state representation [5], which is constructed such that only the unperturbed propagators are affected by the noncommutativity. As a consequence, Feynman diagrams are not separated into planar and nonplanar parts and, in general, all amplitudes are ultraviolet finite (some recent applications of this method are in [6]). We stress that this formalism is unrelated to the ordering prescriptions inherent to the Weyl correspondence in canonical noncommutative field theories, a point which we clarify in the appendix.

It has been also argued that, under some simple assumptions on the noncommutativity matrix, Lorentz preserving noncommutative models may be constructed on even spacetime dimensions, using coherent states. In our case, because the spacetime dimension is odd, Lorentz symmetry is being explicitly broken. However, we may envisage the possibility that the breaking occurs only at very high energies so that its net effect is strongly suppressed at our energy scale. We will show that this is indeed the case in this model. One may entertain the hope that the same mechanism may work for more realistic field theories operating in four-dimensional spacetime [7].

This work is organized as follows. The commutative Gross–Neveu model and its canonical noncommutative counterpart are described in section 2. Using a coherent state approach, in section 3, we show that the problems in the canonical approach are circumvented. In this context we describe the main properties of the model and discuss the problem of Lorentz violations. Our conclusions are contained in section 4. The appendix contains an analysis of the various ordering prescriptions used in the canonical noncommutative field theories, and how they may relate (or not) to our approach.

2. The Gross–Neveu model and the canonical noncommutativity

The commutative Gross–Neveu model is specified by the Lagrangian density

$$\mathcal{L} = \frac{\mathrm{i}}{2}\overline{\psi}\,\partial\!\!\!/\psi - \frac{\sigma}{2}(\overline{\psi}\psi) - \frac{N}{4g}\sigma^2, \tag{1}$$

where ψ_i, $i = 1, \ldots, N$, are two-component Majorana fields and σ is an auxiliary field (note that the replacement of the σ field's equation of motion in equation (1) leads to the usual four-fermion interaction). At the quantum level, it is convenient to replace σ by $\sigma + M$ where M is the vacuum expectation value of the original σ. The new Lagrangian is

$$\mathcal{L} = \frac{\mathrm{i}}{2}\overline{\psi}\,\partial\!\!\!/\psi - \frac{M}{2}\overline{\psi}\psi - \frac{\sigma}{2}(\overline{\psi}\psi) - \frac{N}{4g}\sigma^2 - \frac{N}{2g}M\sigma. \tag{2}$$

Observe now that the coupling constant renormalization, $1/g \rightarrow 1/g_R + \Delta$, where g_R is the renormalized coupling constant, equally affects the tadpole and the two-point vertex function of the auxiliary field. In fact, since the σ field now has zero vacuum expectation value, the gap equation

$$\frac{M}{2g} - \mathrm{i}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{M}{k^2 - M^2} = 0, \tag{3}$$

must be obeyed. Now, the computation of the two-point vertex function of the σ field leads to

$$\begin{aligned}\Gamma_\sigma^{(2)} &= -\frac{\mathrm{i}N}{2g} - N\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{k\cdot(k+p)+M^2}{(k^2-M^2)[(k+p)^2-M^2]}\\ &= -\frac{\mathrm{i}N}{2g} + N\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2-M^2}\\ &\quad + \frac{(p^2-4M^2)N}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(k^2-M^2)[(k+p)^2-M^2]},\end{aligned} \tag{4}$$

which shows that the replacement $1/g \to 1/g_R + \Delta$ eliminates divergences both in the gap equation and in the propagator for the auxiliary field.

We now consider the extension of the above model to a noncommutative space characterized by the commutation relation between coordinate operators

$$[\hat{q}^\mu, \hat{q}^\nu] = \mathrm{i}\Theta^{\mu\nu} \tag{5}$$

and set $\Theta^{\mathrm{i}0} = 0$ to keep time local, thus avoiding unitarity/causality problems [8]. We also adopt the notation $\Theta^{ij} = \mathrm{i}\varepsilon^{ij}\Theta$ where ε^{ij} is the Levi-Cività anti-symmetrical symbol.

In the Weyl–Moyal approach to noncommutative field theories, the pointwise multiplication of fields is replaced by the Moyal product between them. For a given model the propagators are the same as in the corresponding commutative model but the vertices are modified by trigonometric factors. As a consequence, in our situation the gap equation remains unchanged whereas the factor $\cos^2(k^\mu p^\nu \Theta_{\mu\nu})$ appears in the integral in the first line of equation (4), thus leading to

$$\begin{aligned}\Gamma_\sigma^{(2)} = &-\frac{\mathrm{i}N}{2g} + \frac{N}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2-M^2} + \frac{(p^2-4M^2)N}{4}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(k^2-M^2)[(k+p)^2-M^2]}\\ &+ \frac{N}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\cos(2k\wedge p)}{k^2-M^2} + \frac{(p^2-4M^2)N}{4}\\ &\times \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\cos(2k\wedge p)}{(k^2-M^2)[(k+p)^2-M^2]}.\end{aligned} \tag{6}$$

The integrals in the second line of the above equation are finite due to the trigonometric factor $\cos(2k\wedge p)$, however, the counterterm Δ, which is fixed by the gap equation, does no longer eliminate the divergent integral in the first line, because of the factor $1/2$ appearing there. The model has become nonrenormalizable!

3. The noncommutative Gross–Neveu model using coherent states

Faced with the problem outlined in the previous section, one could try to find whether alternative approaches to spacetime noncommutativity could be used to modify the Gross–Neveu model without spoiling the delicate equilibrium between the renormalization of the gap equation and the auxiliary field propagator. One possibility is to use the coherent state approach proposed in [5]. In this case, the commutation relation (5) between the coordinates $\hat{q}^1$ and $\hat{q}^2$ implies that the complex variable $\hat{z} = \frac{\hat{q}^1 + \mathrm{i}\hat{q}^2}{\sqrt{2}}$ and its complex conjugate $\hat{z}^\dagger$ satisfy

$$[\hat{z}, \hat{z}^\dagger] = \Theta. \tag{7}$$

Defining a 'vacuum' state through

$$\hat{z}|0\rangle = 0 \qquad \langle 0|\hat{z}^\dagger = 0, \tag{8}$$

we may construct eigenstates of the 'number' operator $\frac{\hat{z}^\dagger\hat{z}}{\Theta}$ by applying powers of the 'creation' operator $\hat{z}^\dagger$ to the vacuum,

$$\frac{\hat{z}^\dagger\hat{z}}{\Theta}(\hat{z}^\dagger)^n|0\rangle = n(\hat{z}^\dagger)^n|0\rangle. \tag{9}$$

Coherent states, which are eigenstates of the annihilation operator $\hat{z}$, i.e., $\hat{z}|\alpha\rangle = \alpha|\alpha\rangle$, are given by

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\exp(\alpha\hat{z}^\dagger)|0\rangle. \tag{10}$$

Introducing commutative coordinates by $\alpha = x + \mathrm{i}y$, with each classical field $f(x)$ the Fourier representation

$$\hat{\Phi}(\hat{q}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k\hat{q}}\,\tilde{\psi}(k), \tag{11}$$

where $\tilde{\psi}(k)$ denotes the Fourier transform of $f(x)$, associates a field operator $\hat{\Phi}(\hat{q})$. The expectation value of this operator defines a classical field

$$\psi(x) = \langle\alpha|\hat{\Phi}(\hat{q})|\alpha\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}kx-\frac{1}{4}\Theta|\vec{k}|^2}\,\tilde{\psi}(k). \tag{12}$$

The above expression defines the coherent representation for the classical field $f(x)$. If $f(x)$ is a quantized free scalar field the propagator for the coherent field $\psi(x)$ is given by

$$\begin{aligned}\Delta_F(x-y) &\equiv \langle 0|T\psi(x)\psi(y)|0\rangle \\ &= \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k_1x-\mathrm{i}k_2y}\,\mathrm{e}^{-\frac{1}{4}\Theta(|\vec{k}_1|^2+|\vec{k}_2|^2)}(2\pi)^3\delta^3(k_1+k_2)\frac{\mathrm{i}}{k_1^2-m^2} \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}k(x-y)}\frac{\mathrm{i}}{k^2-m^2}\,\mathrm{e}^{-\frac{1}{2}\Theta|\vec{k}|^2}.\end{aligned} \tag{13}$$

In the following we are going to formulate interacting field theories as the quantum versions of the classical fields defined above. In a more precise term, the Lagrangian density for a self-interacting field ψ reads

$$\mathcal{L}(x) = \psi(x)\,\mathrm{e}^{-\frac{\Theta}{2}\vec{\nabla}^2}\mathcal{O}\psi(x) + \mathcal{L}_{\text{int}}(x), \tag{14}$$

where $\mathcal{O} = -(\partial_\mu\partial^\mu + m^2)$ or $\mathcal{O} = (\mathrm{i}\not{\partial} - M)$ for scalar or fermionic fields, respectively. The interacting Lagrangian density $\mathcal{L}_{\text{int}}(x)$ is a polynomial in the basic field and its derivatives. Note that the extra nonlocal factor in the free part of the Lagrangian was devised so as to reproduce the result (13). However, one chooses the interacting Lagrangian as some local product of the basic field $\psi(x)$.

In the case of the Gross–Neveu model each field is replaced by its corresponding field representative, using the correspondence in equation (11), so that the interaction vertices look the same as in the commutative situation. The computation of the gap equation now leads to

$$\frac{1}{2g} - \mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2-M^2}\,\mathrm{e}^{-\frac{1}{2}\Theta|\vec{k}|^2} = 0. \tag{15}$$

and we obtain

$$\frac{1}{g} = \frac{\mathrm{e}^{\frac{\Theta}{2}M^2}}{2\sqrt{2\pi\Theta}}\,\mathrm{Erfc}\left[M\sqrt{\frac{\Theta}{2}}\right] = \frac{1}{2\sqrt{2\pi\Theta}} - \frac{M}{2\pi} + \mathcal{O}(\Theta) \tag{16}$$

where $\mathrm{Erfc}[z] = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t$ denotes the complementary error function and the last equality indicates the leading behaviour of the left-hand side for small Θ. The gap equation (15) is finite for non-zero Θ, and can be made regular in the $\Theta \to 0$ limit by means of a coupling constant renormalization $\frac{1}{g} \to \frac{1}{g_R} - \frac{1}{2\sqrt{2\pi\Theta}}$.

Let us now consider the propagator for the auxiliary field σ, $\Delta_\sigma = -\left[\Gamma_\sigma^{(2)}\right]^{-1}$, where

$$\Gamma_\sigma^{(2)}(p) = \frac{\mathrm{i}N}{2g}\,\mathrm{e}^{\frac{\Theta}{2}|\vec{p}|^2} - \Sigma_\sigma(p) \tag{17}$$

and

$$\begin{aligned}\Sigma_\sigma(p) &= -N\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k\cdot(k+p)+M^2}{(k^2-M^2)[(k+p)^2-M^2]}\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}|^2}\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}+\vec{p}|^2}\\ &= -N\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}|^2}\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}+\vec{p}|^2}}{k^2-M^2}\\ &\quad+\frac{(p^2-4M^2)N}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}|^2}\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}+\vec{p}|^2}}{(k^2-M^2)[(k+p)^2-M^2]}.\end{aligned} \tag{18}$$

Note that all integrals are finite, the integrands being exponentially damped as the loop momenta increase. In this formalism, the noncommutativity of spacetime manifests itself in the appearance of an effective regularization of the loop integrals. It must be stressed, however, that the interpretation of (18) is not that of a regularized Feynman integral, as in usual quantum field theory, since the 'cutoff' $1/\Theta$ is a natural scale which is not introduced as an intermediate step in the renormalization procedure. In this context, the scale Θ is in principle small, but *finite*.

We may be interested in studying the commutative limit of our model; however, we find that it is not possible to get a smooth $\Theta \to 0$ limit. This is so because the leading behaviour for small Θ in equation (18) is different from the one in equation (16). As in the canonical noncommutativity approach, the delicate equilibrium between the gap equation and the auxiliary field propagator renormalizations is lost but, at least, here the problem appears only if one insists in having a smooth $\Theta \to 0$ limit. Whenever the noncommutativity parameter is kept finite, the coherent states approach is able to keep all divergences under control. Also, despite the non-analiticity in Θ, there is no UV/IR mixing present in this approach, since all integrals are regular for vanishing external momentum. With the coupling constant renormalization previously adopted, $\Delta_\sigma \to 0$ leading to a peculiar theory as $\Theta \to 0$.

Alternatively, one could use the coupling constant renormalization to eliminate the divergent integral in equation (18), at the price of a $1/\sqrt{\Theta}$ singularity in the gap equation (15), implying in the vanishing of the renormalized coupling constant in the $\Theta \to 0$ limit (characterizing an asymptotically free model). Again, this is in contrast with the UV/IR problem in canonical noncommutative theories, which threatens the renormalization program because of the blow up of higher order quantum corrections [9].

We turn our attention now to the ψ field two-point function, whose leading correction is given by

$$\Sigma(p) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\not{k}+\not{p}+M}{(k+p)^2-M^2}\Delta_\sigma(k)\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}+\vec{p}|^2}. \tag{19}$$

This expression is clearly well defined as far as $\Theta \neq 0$. Nonetheless, it linearly diverges as $\Theta \to 0$ so that, to get a smooth limit, we can renormalize the model by imposing that the ψ

field propagator satisfies

$$[\Delta_\psi(p)]^{-1} \overset{p\to 0}{\approx} -\mathrm{i}(\not{p} - m) + \mathcal{O}(p^2). \tag{20}$$

Another aspect that raises concern is the Lorentz violation (LV) embodied in the commutation relation (5). Indeed, several authors have pointed out the difficulties in conciliating the LV induced in canonical noncommutative field theories with the known experimental constraints [7, 10, 11], and this has motivated the search for Lorentz-preserving noncommutative models [12].

As far as the coherent state approach is concerned, it was claimed that in even-dimensional spacetime it is possible to avoid the LV by a clever choice of the noncommutativity matrix $\Theta_{\mu\nu}$ [5]. In opposition to this result, in odd spacetime dimensions as is our case, the use of the coherent state basis inevitably leads to a LV. However, one may argue that, if the Green functions of the basic field ψ can be made analytical in Θ, the breaking is necessarily small for small Θ. We can explicitly check this for the two-point function in equation (19).

From a theoretical standpoint, the parameter

$$\xi[\Pi(p)] = \left[\left(\frac{\partial^2}{\partial(p^0)^2} + \frac{\partial^2}{\partial(p^1)^2}\right)\Pi(p)\right]_{p=0}, \tag{21}$$

suggested in [11] was used to measure the LV in the scalar amplitude $\Pi(p)$. That ξ is an adequate measure follows from the fact that it always vanishes if $\Pi(p)$ is Lorentz invariant, while ξ differs from zero in the Lorentz-violating case (ξ corresponds to a Lorentz-violating correction to the dispersion relation of the scalar particle).

As the basic field of the Gross–Neveu model is a spinor, some modification is necessary and we propose the use of

$$\chi[\Sigma(p)] = \left[\left(\frac{\partial}{\partial(p^0)} + \frac{1}{2}\sum_{i=1}^{2}\gamma_i\frac{\partial}{\partial(p^i)}\right)\Sigma(p)\right]_{p=0}, \tag{22}$$

as a measure of the LV in the fermion self-energy $\Sigma(p)$. One can check that $\chi = 0$ in a Lorentz invariant theory. By applying this differential operator to (19) we obtain

$$\chi = \chi^{(0)} + \Theta\chi^{(1)}, \tag{23}$$

where

$$\chi^{(0)} = -2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\gamma^0k^0 + \frac{1}{2}\sum_{i=1}^{2}\gamma^ik^i\right)\frac{\not{k}+M}{(k^2-M^2)^2}\Delta_\sigma(k)\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}^2|} \tag{24}$$

and

$$\chi^{(1)} = \frac{1}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\sum_{i=1}^{2}\gamma^ik^i\right)\frac{\not{k}+M}{k^2-M^2}\Delta_\sigma(k)\,\mathrm{e}^{-\frac{\Theta}{2}|\vec{k}^2|}. \tag{25}$$

Habitually, we conjecture that Θ is very small, being of the order of two powers of the Planck length. From this perspective, $\chi^{(1)}$ is a very small effect of the LV, but the presence of the first term, $\chi^{(0)}$, may appear troublesome at first sight. Such worries are unfounded since, to enforce the renormalization condition (20), one has to replace $\Sigma(p)$ by

$$\Sigma_R(p) = \Sigma(p) - \Sigma(0) - p^\mu\left[\frac{\partial}{\partial(p^\mu)}\Sigma(p)\right]_{p=0} \tag{26}$$

and it is easily found that $\chi[\Sigma_R(p)] = 0$ so that a large Lorentz violation does not appear. We would like to stress that, in the canonical approach to noncommutativity, i.e., by use of the Moyal product, the above procedure is not available as the planar parts of Feynman amplitudes are in general not renormalizable. This is a clear advantage of the coherent state approach.

4. Conclusions

In this paper, we have shown that the use of a coherent states approach for the introduction of spacetime noncommutativity avoids serious problems with the canonical noncommutative extension of the Gross–Neveu model. In this last context, the fact that part of the original ultraviolet divergences survive, and that a single coupling constant renormalization is available to eliminate divergences in two very different structures, makes the theory non-renormalizable. However, in the coherent states formalism, one evades such troubles, and the resulting noncommutative Gross–Neveu model is finite and free of UV/IR mixing for non-vanishing noncommutativity parameter Θ.

We have also studied the generation of Lorentz-violating corrections to the dispersion relation of the model. For finite Θ, if we do not perform any subtraction on the Green functions, large Lorentz violation does appear. This problem can be surmounted if we insist that our model should have a well-behaved $\Theta \to 0$ limit. In this case, a renormalization procedure must be implemented, and this takes care of the LV. Curiously, the $\Theta \to 0$ limit is not the commutative Gross–Neveu model, but an asymptotically free theory.

In this work, we adopted the idea that the 'blurring' effect of the noncommutativity of coordinates, which would be induced by quantum gravity [16], is completely embodied in the Fourier transform of a single field, as in equation (12). This induces the modified propagators that we used. As for the interaction part, we choose the simplest possibility, which is a local product of the classical field defined by equation (12). There are certainly more complicated choices, yet even this simple possibility has interesting implications, as our analysis of the Gross–Neveu model indicates.

Acknowledgments

This work was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES). The work of AFF was supported by FAPESP, project 04/13314-4.

Appendix. On the question of the ordering prescription

The canonical noncommutativity approach is based on a correspondence between classical functions and quantum operators, which fixes the form of the Moyal product used to define noncommutative models. This correspondence is defined up to an arbitrary ordering prescription. To take this into account, we introduce a generalized Weyl correspondence [15],

$$\hat{\Phi}^{(f)}(\hat{q}) \equiv \int \mathrm{d}^2x\, \phi(x)\Delta^{(f)}(\hat{q} - x), \tag{A.1}$$

where

$$\Delta^{(f)}(\hat{q} - x) = \int \frac{\mathrm{d}^2k}{(2\pi)^2} f(k)\, \mathrm{e}^{-\mathrm{i}k(\hat{q}-x)}, \tag{A.2}$$

and $f(k) = f(k_1, k_2)$ is an arbitrary function encoding the ordering ambiguity in the relation between functions $\phi(x)$ and operators $\hat{\Phi}^{(f)}(\hat{q})$. One should only requires that f nowhere vanishes and that $f(0) = 1$. Popular ordering choices, such as normal ordering, Weyl ordering and so on, can be implemented by particular choices of $f(k)$; in particular, $f(k) = 1$ yields the usual Weyl correspondence.

The inverse correspondence is given by

$$\phi(x) = \mathrm{Tr}[\hat{\Phi}^{(f)}(\hat{q})\Delta^{(\tilde{f})}(\hat{q} - x)], \tag{A.3}$$

where $\tilde{f}(k) = 1/f(-k)$. Here, the trace is normalized as $\mathrm{Tr}(\mathrm{e}^{-\mathrm{i}k\hat{q}}) = (2\pi)^2\delta^2(k)$. From this inverse map, one defines a star-product involving n classical functions,

$$\begin{aligned}\phi_1(x) \star \phi_2(x) \star \cdots \star \phi_n(x) &\equiv \mathrm{Tr}\big[\hat{\Phi}_1^{(f)}(\hat{q})\hat{\Phi}_2^{(f)}(\hat{q}) \cdots \hat{\Phi}_n^{(f)}(\hat{q})\Delta^{(\tilde{f})}(\hat{q} - x)\big] \\ &= \int \left[\prod_{i=1}^{n} \frac{\mathrm{d}^2 k_i}{(2\pi)^2}\right]\left[\prod f(k_i)\right] \tilde{f}\left(-\sum k_i\right) \mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{i<j} k_i \wedge k_j} \\ &\quad\times \mathrm{e}^{-\mathrm{i}(\sum_i k_i)x} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \cdots \tilde{\phi}_n(k_n),\end{aligned} \tag{A.4}$$

where $\tilde{\phi}_i(k_i)$ is the Fourier transform of $\phi_i(x)$. Note that all integrals are two-dimensional since there are two noncommuting coordinates $\hat{q}_1$ and $\hat{q}_2$, time being a commutative parameter untouched by the correspondence. Thus, the spacetime integral of (A.4) can be cast as

$$\begin{aligned}\int \mathrm{d}^3x\, \phi_1(x) \star \phi_2(x) \star \cdots \star \phi_n(x) &= \int \left[\prod_{i=1}^{n} \frac{\mathrm{d}^3 k_i}{(2\pi)^3}\right](2\pi)^3\delta^3\left(\sum_i k_i\right) \\ &\quad\times f(k_1)f(k_2)\cdots f(k_n)\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{i<j} k_i\wedge k_j} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\cdots\tilde{\phi}_n(k_n).\end{aligned} \tag{A.5}$$

As becomes clear from equation (A.4), the usual Moyal product is obtained for all f's satisfying $f(k+q) = f(k)f(q)$, which obviously happens for $f = 1$ but not for other popular orderings, for example the normal ordering, which is reproduced by

$$f(k) = \exp\left(\frac{\theta}{2}|\vec{k}^2|\right). \tag{A.6}$$

Typically, both propagators and vertices will be modified by the f factors present in (A.5), but these modifications will disappear when one calculates the quantum corrections to the effective action of the theory, and the result will be the same as the one in the usual Moyal-product approach. Indeed, for the quadratic part of the action, one has

$$\int \mathrm{d}^3x\, \phi_1(x) \star \mathcal{O}\phi_1(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} f(k)f(-k)\tilde{\phi}_1(k_1)\tilde{\mathcal{O}}\tilde{\phi}_2(k_2), \tag{A.7}$$

so that internal propagators acquire a $1/f^2$ factor. However, this $1/f^2$ cancels the f's arising from the vertices attached to the ends of the internal lines. In this way, even if the Moyal product is sensible to the ordering choice, the quantum theory seems to be actually independent of the operator ordering [13, 14]. One could allow for different orderings for the free and interactions parts of the Lagrangian [13] but, even in this case, the 'standard' Moyal factor $\exp\left(-\frac{\mathrm{i}}{2}\sum_{i<j} k_i \wedge k_j\right)$ would be present. This is essentially different from the coherent state approach where the damping exponentials in the free propagators are not cancelled in the computation of general Feynman amplitudes, and the 'standard' Moyal factor is absent. This happens because the noncommutativity is already embodied in the Fourier transform of a single field, as explicitly shown in equation (12), and the interaction Lagrangian is chosen to be a simple, local, product of fields.

References

[1] See for instance Douglas M and Nekrasov N A 2001 *Rev. Mod. Phys.* **73** 977
Szabo R 2003 *Phys. Rep.* **378** 207
Gomes M 2002 *Proc. XI Jorge André Swieca Summer School, Particles and Fields* ed G A Alves, O J P Éboli and V O Rivelles (New York: World Scientific)
Girotti H O 2003 Noncommutative quantum field theories *Preprint* hep-th/0301237
[2] Campbell B A and Kaminsky K 2001 *Nucl. Phys.* B **606** 613
Petriello F J 2001 *Nucl. Phys.* B **601** 169
[3] Akhmedov E T, DeBoer P and Semenoff G W 2001 *J. High Energy Phys.* JHEP06(2001)009 JHEP06(2001)009
Akhmedov E T, DeBoer P and Semenoff G W 2001 *Phys. Rev.* D **64** 065005
[4] Girotti H O, Gomes M, Rivelles V O and da Silva A J 2002 *Int. J. Mod. Phys.* A **17** 1503
Girotti H O, Gomes M, Petrov A Yu, Rivelles V O and da Silva A J *Phys. Lett.* B **521** 119
[5] Smailagic A and Spallucci E 2003 *J. Phys. A: Math. Gen.* **36** L517
Smailagic A and Spallucci E 2003 *J. Phys. A: Math. Gen.* **36** L467
Smailagic A and Spallucci E 2004 *J. Phys. A: Math. Gen.* **37** 7169
[6] Jahan A 2005 *Phys. Lett.* B **623** 179
Casadio R, Cox P H, Harma B and Micu O 2006 *Phys. Rev.* D **73** 044019
Anacleto M A, Nascimento J R and Petrov A Yu 2006 *Phys. Lett.* B **637** 344
[7] Ferrari A F, Girotti H O and Gomes M 2006 *Phys. Rev.* D **73** 047703
[8] Gomis J and Mehen T 2000 *Nucl. Phys.* B **591** 265
[9] Minwalla S, van Raamsdonk M and Seiberg N 2000 *J. High Energy. Phys.* JHEP02(2000)020 JHEP02(2000)020
[10] Carlson C E, Carone C D and Lebed R F 2001 *Phys. Lett.* B **518** 201
Carlson C E, Carone C D and Lebed R F 2002 *Phys. Lett.* B **549** 337
Anisimov A, Banks T, Dine M and Graesser M 2002 *Phys. Rev.* D **65** 085032
[11] Collins J, Perez A, Sudarsky D, Urrutia L and Vucetich H 2004 *Phys. Rev. Lett.* **93** 191301
[12] Carlson C E, Carone C D and Zobin N 2002 *Phys. Rev.* D **66** 075001
[13] Chaichian M, Demichev A and Presnajder P 2000 *Nucl. Phys.* B **567** 360
[14] Hammou A B, Lagraa M and Sheikh-Jabbari M M 2002 *Phys. Rev.* D **66** 025025
[15] Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2161
[16] Doplicher S, Fredenhagen K and Roberts J E 1995 *Commun. Math. Phys.* **172** 187